A Comparative Study of Stability Methods for Flexible Satellites

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This paper compares three approaches to the stability of hybrid dynamical systems, all three methods being based on the Liapunov direct method. The first method uses testing density functions, whereas the second involves defining certain integral coordinates. Both the method using testing density functions and the method of integral coordinates lead to closed-form stability criteria in terms of the system parameters. Criteria obtained using the method of integral coordinates are, in general, less restrictive than those derived by the method using testing density functions. On the other hand, the latter method is easier to apply and requires less work than the former. The third method is the standard modal analysis. The modal analysis generally yields more involved criteria, depending on the number of modes used to represent the elastic displacements. As an application, the attitude stability of an Earth-pointing satellite with multielastic domains is investigated.

Introduction

THE motion of satellites with distributed flexible parts can be described by a set of ordinary differential equations for the rotational motion of a given reference frame and a set of partial differential equations for the elastic motion relative to that frame.[1] Such a system of differential equations of motion has come to be known as a hybrid dynamical system.

A common approach to the stability of distributed-parameter systems is to discretize them. The most frequently used approach is to regard the system properties as concentrated at certain points of the domain of extension of the elastic continuum. This approach is ordinarily referred to as the "lumped parameter method" and less often as "spatial discretization." Another common procedure is to represent dependent variables describing the motion of the continuum by finite series of space-dependent eigenfunctions multiplied by time-dependent generalized coordinates. Certain weighted integrations over the domain of extension of the continuum eliminate the space dependence, leading to a multidegree-of-freedom discrete system. This method is generally known as "modal analysis" and less frequently as "modal truncation." In an early attempt to treat rigorously systems with elastic parts, Meirovitch and Nelson[2] used both these techniques to analyze the stability of motion of a spinning satellite with flexible rods along the spin axis. Reference 2 also contains a first study of the effect of modal truncation on the stability criteria. In a later work. Dokuchaev[3] used a single-mode approximation, in conjunction with the modal analysis, to study the stability of motion of a spinning satellite with five elastic rods. The main criticism of the modal analysis is the uncertainty involved in the effect of the series truncation on the stability criteria derived. Moreover, for a certain system there are no clues as to the number of modes necessary for the method to yield satisfactory results. Hence, an in-depth study of the effect of series truncation is highly desirable.

In a first attempt to apply Liapunov's direct method to hybrid systems from the area of satellite dynamics, Meirovitch[1,4] studied the stability of a spinning rigid body with elastic appendages. Several new concepts were introduced in Ref. 1, such as the use of the bounding properties of Rayleigh's quotient to eliminate the dependence of the testing functional on the spatial derivatives, as well as the concept of a testing density function. Reference 5 extends the development of Refs. 1 and 4 to hybrid systems with multielastic domains, such as a spinning rigid body with n rigidly-attached flexible appendages simulating an orbiting satellite. The mathematical formulation consists of a hybrid set of $6(n+1)$ Hamiltonian equations of motion, six of which are ordinary differential equations for the rotational motion and $6n$ partial differential equations for the elastic motion. The stability analysis follows the pattern of Ref. 1, in which it is shown that under certain circumstances the system Hamiltonian H is a suitable Liapunov functional. The problem is shown to simplify considerably in the case in which it is possible to define density functions. However, it should be pointed out that this is not always possible.

Another, not so common, discretization scheme, based on the mathematical formulation of Ref. 5, involves the definition of new generalized coordinates representing certain integrals appearing in the testing functional as well as the use of Schwarz's inequality for functions. The method, referred to as "the method of integral coordinates" has been first introduced for the type of problems discussed here in Refs. 6 and 7. A somewhat similar approach, applicable to certain problems involving the stability of bodies with cavities filled with fluids, has been presented in Ref. 8.

This paper uses the formulation of Ref. 5 and presents a comparative study of the application of the method using testing density functions, the method of integral coordinates, and the modal analysis to the stability analysis of hybrid dynamical systems. In particular, the problem of attitude stability of an earth-pointing satellite consisting of a main rigid body and six flexible rods simulating antennas is solved.

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Problem Formulation

The mathematical formulation, including the stability analysis for a hybrid dynamical system with multielastic domains, is essentially that given in Ref. 5. It will not be repeated here, but only the main features summarized.

Let us consider a body consisting of a rigid part occupying the domain D_0 and n flexible parts, rigidly attached to D_0 and occupying the domains D_i $(i = 1, 2, \ldots, n)$ when in undeformed state. The domains D_i have common boundaries only with D_0 (see Fig. 1). Denoting by D the domain of extension of the entire body, we have

$$D = \sum_{i=0}^{n} D_i$$

Let m_i $(i = 0, 1, \ldots, n)$ be the masses associated with the domains D_i, so that the total mass of the system is

$$m = \sum_{i=0}^{n} m_i$$

We shall be concerned with the motion of m in a central-force gravitational field produced by a spherically symmetric body of total mass M, where the center of M will be regarded as the origin of an inertial space XYZ. Hence, we can interpret the motion of m as taking place about a fixed center of force C.F. coinciding with the origin of the space XYZ. Let 0 be the origin of a system xyz, such that 0 is the mass center of m and axes xyz are the principal axes of m when in undeformed state. Note that system xyz is embedded in D_0 but may not be a set of principal axes for D_0. Let us also define sets of axes $x_i y_i z_i$ fixed relative to D_i $(i = 1, 2, \ldots, n)$ and with directions such that the elastic deformations are parallel to these axes for each domain. When the body deforms the mass center of m no longer coincides with 0 in general. We shall denote the mass center of the body in deformed state by c and sets of axes parallel to xyz and $x_i y_i z_i$ but with the origin at c instead of 0 by $\xi\eta\zeta$ and $\xi_i \eta_i \zeta_i$, respectively.

Next denote the radius vector from 0 to a given point in D_i $(i = 0, 1, \ldots, n)$ by $\mathbf{r}_i$, where the point in question coincides with the position of a mass element dm_i when the body is in undeformed state. Denoting by $\mathbf{i}_i$, $\mathbf{j}_i$, $\mathbf{k}_i$ the unit vectors along axes x_i, y_i, z_i, respectively, the position vector $\mathbf{r}_i$ can be written as

$$\mathbf{r}_i = x_i \mathbf{i}_i + y_i \mathbf{j}_i + z_i \mathbf{k}_i \quad (i = 0, 1, \ldots, n)$$

The elastic displacement vector $\mathbf{u}_i$ of element dm_i depends on that position, as well as time, so that

$$\mathbf{u}_i = u_i(x_i, y_i, z_i, t)\mathbf{i}_i + v_i(x_i, y_i, z_i, t)\mathbf{j}_i + w_i(x_i, y_i, z_i, t)\mathbf{k}_i \quad (i = 1, 2, \ldots, n)$$

The displacements u_i, v_i, w_i, measured along axes x_i, y_i, z_i, respectively, *are regarded as infinitesimally small.* Denoting by $\mathbf{r}_c$ the vector from 0 to c, we conclude that $\mathbf{u}_{ci} = \mathbf{u}_i - \mathbf{r}_c$ represents the displacement vector of the element of mass dm_i relative to center c.

Omitting several intermediate steps, which are given in Ref. 5, we can write the kinetic energy in the matrix form

$$T = \tfrac{1}{2} m \{\dot{R}_c\}^T \{\dot{R}_c\} + \tfrac{1}{2} \sum_{i=0}^{n} \{\omega\}^T [l_i]^T [J_i] [l_i] \{\omega\} + \{\omega\}^T \sum_{i=1}^{n} \int_{m_i} [r_i^{(0)} + u_{ci}^{(0)}] [l_i]^T \{\dot{u}'_{ci}\} \, dm_i + \tfrac{1}{2} \sum_{i=1}^{n} \int_{m_i} \{\dot{u}'_{ci}\}^T \{\dot{u}'_{ci}\} \, dm_i \qquad (1)$$

where $\{\dot{R}_c\}$ is a column matrix whose elements represent the velocity components of the mass center c, $\{\omega\}$ the angular velocity matrix of axes $\xi\eta\zeta$ relative to an inertial space, and $[l_i]$ the square matrix of direction cosines between $\xi_i \eta_i \zeta_i$ and $\xi\eta\zeta$. Moreover, $[J_i]$ is the inertia dyadic of m_i in deformed state expressed in terms of components about

$$\xi_i \eta_i \zeta_i, \ [r_i^{(0)} + u_{ci}^{(0)}]$$

a skew-symmetric matrix whose elements are given by the relation

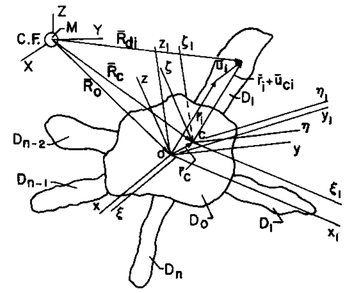

Fig. 1 The rigid body with flexible appendages.

$$r_{inm}^{(0)} + u_{cinm}^{(0)} = \sum_{l=1}^{3} \varepsilon_{mnl} (r_{il}^{(0)} + u_{cil}^{(0)}),$$

in which the superscript 0 indicates the base, ε_{mnl} is the epsilon symbol (see Ref. 9, p. 109), and $\{\dot{u}'_{ci}\}$ the matrix of elastic velocities of dm_i relative to c obtained by letting $\{\omega\} = \{0\}$. Also from Ref. 5, we can write the gravitational potential energy

$$V_G = -\frac{Km}{R_c} - \frac{K}{2R_c^3} \sum_{i=0}^{n} \operatorname{tr}([l_i]^T [J_i] [l_i]) + \frac{3K}{2R_c^3} \sum_{i=0}^{n} \{l_a\}^T [l_i]^T [J_i] [l_i] \{l_a\} \qquad (2)$$

where $K = GM$, in which G is the universal gravitational constant, R_c the magnitude of the vector $\mathbf{R}_c$ (see Fig. 1), and $\{l_a\}$ the matrix of direction cosines between $\mathbf{R}_c$ and $\xi\eta\zeta$. (Note that tr denotes the trace of the matrix in question.) Assuming that the components of displacement corresponding to one elastic domain are not affected by those corresponding to other elastic domains, the elastic potential can be written in the general functional form

$$V_{EL} = \sum_{i=1}^{n} V_{ELi} = \sum_{i=1}^{n} \int_{D_i} \hat{V}_{EL}^{(i)} \left(\frac{\partial u_{ci}}{\partial x_i}, \frac{\partial u_{ci}}{\partial y_i}, \frac{\partial u_{ci}}{\partial z_i}, \frac{\partial v_{ci}}{\partial x_i}, \ldots, \frac{\partial^2 u_{ci}}{\partial x_i^2}, \frac{\partial^2 u_{ci}}{\partial x_i \partial y_i}, \ldots, \frac{\partial^2 w_{ci}}{\partial z_i^2} \right) dD_i \qquad (3)$$

where $\hat{V}_{EL}^{(i)}$ is the elastic potential energy density at a given point of domain D_i.

Assuming that the mass center c moves in a circular orbit, we have $K = \Omega^2 R_c^3$, where Ω and R_c are constants representing the orbital angular velocity and orbit radius, respectively. Moreover, letting abc be an orbital set of axes, as shown in Fig. 2, the orientation of $\xi\eta\zeta$ relative to abc is defined by three rotations θ_1, θ_2, θ_3. The angular velocity matrix $\{\omega\}$ can be regarded as a function of the angular coordinates θ_j and their rates of change $\dot{\theta}_j (j = 1, 2, 3)$. In view of the above, and recalling Eqs. (1–3), we can write the system Lagrangian in the general functional form

$$L = T - V_G - V_{EL} = L^{(0)}(\theta_1, \theta_2, \ldots, \dot{\theta}_3) + \sum_{i=1}^{n} \int_{D_i} \hat{L}^{(i)} \left(\theta_1, \theta_2, \ldots, \dot{\theta}_3, u_{ci}, v_{ci}, \ldots, \dot{w}_{ci}, \frac{\partial u_{ci}}{\partial x_i}, \frac{\partial u_{ci}}{\partial y_i}, \ldots, \frac{\partial w_{ci}}{\partial z_i}, \frac{\partial^2 u_{ci}}{\partial x_i^2}, \frac{\partial^2 u_{ci}}{\partial x_i dy_i}, \ldots, \frac{\partial^2 w_{ci}}{\partial z_i^2} \right) dD_i \qquad (4)$$

where

$$L^{(0)} = T^{(0)} - V_G^{(0)} = \tfrac{1}{2} \{\omega\}^T [J_0] \{\omega\} + \tfrac{1}{2} \Omega^2 \operatorname{tr} [J_0] - \tfrac{3}{2} \Omega^2 \{l_a\}^T [J_0] \{l_a\} \qquad (5a)$$

is the portion of the Lagrangian associated with the rigid part, and

$$\hat{L}^{(i)} = \hat{T}^{(i)} - \hat{V}_G^{(i)} - \hat{V}_{EL}^{(i)} = \tfrac{1}{2} \{\omega\}^T [l_i]^T [\hat{J}_i] [l_i] \{\omega\} + \rho_i \{\omega\}^T [r_i^{(0)} + u_{ci}^{(0)}] [l_i]^T \{\dot{u}'_{ci}\} + \tfrac{1}{2} \rho_i \{\dot{u}'_{ci}\}^T \{\dot{u}'_{ci}\} + \tfrac{1}{2} \Omega^2 \operatorname{tr}([l_i]^T [\hat{J}_i] [l_i]) - \tfrac{3}{2} \Omega^2 \{l_a\}^T [l_i]^T [\hat{J}_i] [l_i] \{l_a\} - \hat{V}_{EL}^{(i)}$$

$$i = 1, 2, \ldots, n \qquad (5b)$$

Fig. 2 The orbital set of axes.

is the Lagrangian density at a given point of the domain D_i, in which $[\hat{J}_i]$ is the inertia matrix density, and ρ_i the mass density at the given point.

Introducing the generalized momenta

$$p_{\theta_j} = \partial L/\partial \dot{\theta}_j, \qquad j = 1,2,3 \tag{6a}$$

and the generalized momentum densities

$$\hat{p}_{u_{ci}} = \frac{\partial \hat{L}^{(i)}}{\partial \dot{u}_{ci}}, \ \hat{p}_{v_{ci}} = \frac{\partial \hat{L}^{(i)}}{\partial \dot{v}_{ci}}, \ \hat{p}_{w_{ci}} = \frac{\partial \hat{L}^{(i)}}{\partial \dot{w}_{ci}}, \quad i = 1,2,\ldots,n \tag{6b}$$

associated with any given point of the domain D_i, the Hamiltonian H can be written in the general functional form

$$\begin{aligned} H &= \sum_{j=1}^{3} p_{\theta_j}\dot{\theta}_j + \sum_{i=1}^{n} \int_{D_i} (\hat{p}_{u_{ci}}\dot{u}_{ci} + \hat{p}_{v_{ci}}\dot{v}_{ci} + \hat{p}_{w_{ci}}\dot{w}_{ci})\,dD_i - L \\ &= H^{(0)}(\theta_1, \theta_2, \ldots, p_{\theta_3}) + \sum_{i=1}^{n} \int_{D_i} \hat{H}^{(i)}\Big(\theta_1, \theta_2, \ldots, p_{\theta_3}, u_{ci}, v_{ci}, \ldots, \\ &\quad \hat{p}_{w_{ci}}, \frac{\partial u_{ci}}{\partial x_i}, \frac{\partial u_{ci}}{\partial y_i}, \ldots, \frac{\partial w_{ci}}{\partial z_i}, \frac{\partial^2 u_{ci}}{\partial x_i^2}, \frac{\partial^2 u_{ci}}{\partial x_i \partial y_i}, \ldots, \frac{\partial^2 w_{ci}}{\partial z_i^2}\Big)\,dD_i \end{aligned} \tag{7}$$

in which the meaning of $H^{(0)}$ and $\hat{H}^{(i)}$ is self-evident.

Following the pattern of Ref. 1, the Hamiltonian equations of motion become

$$\dot{\theta}_j = \frac{\partial H}{\partial p_{\theta_j}}, \ \dot{p}_{\theta_j} = -\frac{\partial H}{\partial \theta_j}, \qquad j = 1,2,3 \tag{8a}$$

$$\begin{aligned} &\dot{u}_{ci} = \frac{\partial \hat{H}^{(i)}}{\partial \hat{p}_{u_{ci}}}, \quad \dot{v}_{ci} = \frac{\partial \hat{H}^{(i)}}{\partial \hat{p}_{v_{ci}}}, \quad \dot{w}_{ci} = \frac{\partial \hat{H}^{(i)}}{\partial \hat{p}_{w_{ci}}} \\ &\dot{\hat{p}}_{u_{ci}} = -\frac{\partial \hat{H}^{(i)}}{\partial u_{ci}} + \mathscr{L}_{u_{ci}}[u_{ci}, v_{ci}, w_{ci}] + \hat{Q}_{u_{ci}} \\ &\dot{\hat{p}}_{v_{ci}} = -\frac{\partial \hat{H}^{(i)}}{\partial v_{ci}} + \mathscr{L}_{v_{ci}}[u_{ci}, v_{ci}, w_{ci}] + \hat{Q}_{v_{ci}} \qquad i = 1,2,\ldots,n \\ &\dot{\hat{p}}_{w_{ci}} = -\frac{\partial \hat{H}^{(i)}}{\partial w_{ci}} + \mathscr{L}_{w_{ci}}[u_{ci}, v_{ci}, w_{ci}] + \hat{Q}_{w_{ci}} \end{aligned} \tag{8b}$$

where Eqs. (8b) must be satisfied at every point of the domain D_i. Moreover, they are subject to the boundary conditions

$$\begin{matrix} \mathbf{B}_1[u_{ci}, v_{ci}, w_{ci}] \cdot \mathbf{B}_3[u_{ci}, v_{ci}, w_{ci}] \\ \mathbf{B}_2[u_{ci}, v_{ci}, w_{ci}] \cdot \mathbf{B}_4[u_{ci}, v_{ci}, w_{ci}] \end{matrix} \quad \text{on } S_i \tag{9a}$$

or

$$\begin{matrix} \mathbf{B}_1[u_{ci}, v_{ci}, w_{ci}] \cdot \mathbf{B}_4[u_{ci}, v_{ci}, w_{ci}] \\ \mathbf{B}_2[u_{ci}, v_{ci}, w_{ci}] \cdot \mathbf{B}_3[u_{ci}, v_{ci}, w_{ci}] \end{matrix} \quad \text{or } S_i \tag{9b}$$

where S_i denotes the surface bounding the domain D_i $(i = 1,2,\ldots, n)$. Note that

$$\mathscr{L}_{u_{ci}}, \ \mathscr{L}_{v_{ci}}, \ \mathscr{L}_{w_{ci}}$$

are differential operators and $\mathbf{B}_k$ $(k = 1,2,3,4)$ are differential operator vectors; the square brackets do not denote matrices in this case. The quantities

$$\hat{Q}_{u_{ci}}, \ \hat{Q}_{v_{ci}}, \ \hat{Q}_{w_{ci}}$$

in Eqs. (8b) designate distributed internal damping forces.

We note that the system of Eqs. (8) is hybrid in the sense that Eqs. (8a) are ordinary and Eqs. (8b) are partial differential equations.

Stability of Hybrid Dynamical Systems

The stability analysis to be used here has been developed in Ref. 5. We present here only the main features. Let us consider a hybrid system with the state vector given by $\mathbf{v} = \mathbf{v}_d(t) + \mathbf{v}_c(P, t)$, where $\mathbf{v}_d(t)$ and $\mathbf{v}_c(P, t)$ represent discrete and continuous variables, respectively. The system is described by the set of differential equations

$$\dot{\mathbf{v}} = \mathbf{V}(\mathbf{v}, \partial\mathbf{v}_c/\partial x, \partial\mathbf{v}_c/\partial y, \ldots, \partial^{2p}\mathbf{v}_c/\partial z^{2p}) \tag{10}$$

where $\mathbf{V}$ is a vector function depending on the state vector and spatial derivatives of the state vector through order $2p$, in which p is an integer. The state vector can be imagined geometrically as representing an element in a space S (see Ref. 5). A solution of system (10) constant in time, namely, a set of constants satisfying

$$\mathbf{V}(\mathbf{v}, \partial\mathbf{v}_c/\partial x, \partial\mathbf{v}_c/\partial y, \ldots, \partial^{2p}\mathbf{v}_c/\partial z^{2p}) = \mathbf{0} \tag{11}$$

is known as a singular point or equilibrium point. Assuming, without loss of generality, that the origin of S is an equilibrium point, we shall concern ourselves with the stability of the trivial, or null solution.

Stability is now defined in a manner analogous to the Liapunov definitions of stability for discrete systems. To this end, we first introduce the norm

$$\|\mathbf{v}(t)\| = \|\mathbf{v}_d(t)\| + \textstyle\int_D \|\mathbf{v}_c(P, t)\|\, dD(P)$$

where D is the domain over which continuous variables are defined, and denote by $\|\mathbf{v}_0\|$ the value of the norm at $t = t_0$. Then the trivial solution is defined as stable if for any arbitrary positive quantitity ε and time t_0 there exists a positive number $\delta(\varepsilon, t_0)$ such that the satisfaction of the inequality $\|\mathbf{v}_0\| < \delta$ implies the satisfaction of the inequality $\|\mathbf{v}(t)\| < \varepsilon$ for all $t \geqq t_0$. If, in addition,

$$\lim_{t\to\infty} \|\mathbf{v}(t)\| = 0$$

then the trivial solution is asymptotically stable. We are concerned exclusively with the cases in which a small initial state $\|\mathbf{v}_0\|$ implies also small spatial derivatives, at least through order p. The trivial solution is unstable if it is not stable.

To test the stability of system (10) in the neighborhood of the trivial solution, we define a scalar functional $U = U(\mathbf{v}, \partial\mathbf{v}_c/\partial x, \partial\mathbf{v}_c/\partial y, \ldots, \partial^p\mathbf{v}_c/\partial z^p)$ such that $U(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) = 0$. We note that U depends on spatial derivatives through order p, as opposed to $\mathbf{V}$ which depends on derivatives through order $2p$. Moreover, the total time derivative of U along a trajectory of the system is defined by

$$\dot{U} = dU/dt = \nabla U_d \cdot \dot{\mathbf{v}}_d + \textstyle\int_D \nabla \hat{U}_c \cdot \dot{\mathbf{v}}_c\, dD = \nabla U_d \cdot \mathbf{V}_d + \textstyle\int_D \nabla \hat{U}_c \cdot \mathbf{V}_c\, dD$$

where the subscripts d and c designate quantities pertaining to discrete and continuous variables, respectively.

Four theorems are presented in Ref. 5 according to which it is possible to determine the system stability, or lack of it, from the sign properties of U. Since the scalar functional U depends on spatial derivatives of $\mathbf{v}$, it may be difficult at times to determine its sign properties. In such cases it may be possible to define another scalar functional $W(\mathbf{v})$, depending on the state vector $\mathbf{v}$ alone, and such that $U \geqq W$. Then we can state the following.[5]

Stability Theorem

Suppose that for system (10) there exists a scalar functional U such that $\dot{U}$ is negative semidefinite along every trajectory of Eq. (10) and, in addition, the set of points at which $\dot{U}$ is zero contains no nontrivial positive half-trajectory. Then, if a positive definite functional W can be found such that $U \geqq W$, the trivial solution $\mathbf{v} = \mathbf{0}$ is asymptotically stable.

The aforementioned Stability Theorem is an extension of certain theorems by Liapunov and Krasovskii.[5] The Stability Theorem has significant implications as far as the stability analysis of hybrid dynamical systems of the type considered here is concerned.

In the case of our hybrid dynamical system the finite dimensional vector space associated with $\mathbf{v}_d$ is defined by θ_j, p_{θ_j} $(j = 1,2,3)$ and the function space associated with

$$\mathbf{v}_c \text{ by } u_{ci}, v_{ci}, w_{ci}, \hat{p}_{u_{ci}}, \hat{p}_{v_{ci}}, \hat{p}_{w_{ci}} \qquad (i = 1,2,\ldots,n)$$

The space S is recognized as simply the phase space, which can alternatively be regarded as being defined by the generalized coordinates and velocities.

Using Eqs. (7) and (8), it can be shown that

$$\dot{H} = \sum_{i=1}^{n} \int_{D_i} (\hat{Q}_{u_{ci}}\dot{u}_{ci} + \hat{Q}_{v_{ci}}\dot{v}_{ci} + \hat{Q}_{w_{ci}}\dot{w}_{ci})\,dD_i \leqq 0 \tag{12}$$

where only such

$$\hat{Q}_{u_{ci}}, \ \hat{Q}_{v_{ci}}, \ \hat{Q}_{w_{ci}} \qquad (i = 1,2,\ldots,n)$$

have been considered that render H negative semidefinite and reduce to zero only at the origin.

A case of special interest is that in which the angular motion is referred to a rotating coordinate frame. In this case the angular velocity vector can be written in the form

$$\{\omega\} = \{\omega\}_0 + \{\omega\}_1 \tag{13}$$

where $\{\omega\}_0$ depends on θ_j $(j = 1,2,3)$ alone, whereas $\{\omega\}_1$ depends on $\theta_j, \dot{\theta}_j$ $(j = 1,2,3)$. In particular, $\{\omega\}_1$ is a linear combination of the angular velocities $\dot{\theta}_j$. Under these circumstances, the Hamiltonian can be shown to have the form

$$H = T_2 - T_0 + V_G + V_{EL} \tag{14}$$

where

$$T_2 = \tfrac{1}{2}\{\omega\}_1^T[J_0]\{\omega\}_1 + \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}\{\omega\}_1^T[l_i]^T[\hat{J}_i][l_i]\{\omega\}_1\, dD_i + \sum_{i=1}^{n}\int_{D_i}\rho_i\{\omega\}_1^T[r_i^{(0)} + u_{ci}^{(0)}][l_i]^T\{\dot{u}'_{ci}\}\, dD_i + \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}\rho_i\{\dot{u}'_{ci}\}^T\{\dot{u}'_{ci}\}\, dD_i \tag{15a}$$

is quadratic in the generalized velocities and

$$T_0 = \tfrac{1}{2}\{\omega\}_0^T[J_0]\{\omega\}_0 + \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}\{\omega\}_0^T[l_i]^T[\hat{J}_i][l_i]\{\omega\}_0\, dD_i \tag{15b}$$

does not contain generalized velocities at all.

Whereas T_2 depends on the generalized coordinates and velocities, T_0 and V_G depend on the generalized coordinates alone. On the other hand, V_{EL} depends on spatial derivatives of the elastic displacements which makes the testing for positive definiteness of the Hamiltonian considerably more involved. There are cases, however, when these difficulties can be circumvented, namely, when in the elastic potential energy the displacements u_{ci}, v_{ci}, and w_{ci} are independent of one another. Then, following the reasoning of Ref. 5, we can use the bounding property of Rayleigh's quotient to show that

$$V_{EL} = \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}(u_i\mathscr{L}_{u_i}[u_i] + v_i\mathscr{L}_{v_i}[v_i] + w_i\mathscr{L}_{w_i}[w_i])\, dD_i \geqq \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}\rho_i\{u_{ci}\}^T[\Lambda_{1i}{}^2]\{u_{ci}\}\, dD_i \tag{16}$$

where $[\Lambda_{1i}{}^2]$ is a diagonal matrix of the lowest eigenvalues associated with the vibrations u_i, v_i, w_i, respectively. Introducing the new functional

$$\kappa = T_2 - T_0 + V_G + \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}\rho_i\{u_{ci}\}^T[\Lambda_{1i}{}^2]\{u_{ci}\}\, dD_i \tag{17}$$

where κ is such that $H \geqq \kappa$, we conclude from the Stability Theorem introduced above that the trivial solution is asymptotically stable if κ is positive definite in the neighborhood of the origin.

But κ can be written as the sum $\kappa = \kappa_1 + \kappa_2$, where $\kappa_1 = T_2$ which is positive definite by definition. It follows that the null solution is asymptotically stable if κ_2 is positive definite. In general $\{\omega\}_0$ involves the orbital angular velocity Ω and can be written as $\{\omega\}_0 = \Omega\{l_c\}$, where $\{l_c\}$ is the column matrix of direction cosines between axis c and axes $\xi\eta\zeta$. Moreover, we wish to write matrix $[J]$ as the sum of $n+1$ parts, namely, $[J]_r$ and $[J_i]_e$ $(i = 1,2,\ldots,n)$, where the first is obtained by regarding the entire body as rigid and the remaining n parts represent changes due to the elastic deformations, so that

$$[J] = [J]_r + \sum_{i=1}^{n}[J_i]_e \qquad [J]_r = [J_0] + \sum_{i=1}^{n}[l_i]^T[J_i]_r[l_i] \tag{18}$$

where $[J_i]_r$ is the inertia matrix of the mass associated with domain D_i when the part is in undeformed state. In view of this, we obtain

$$\kappa_2 = -\tfrac{1}{2}\Omega^2(\{l_c\}^T[J]_r\{l_c\} + \mathrm{tr}\,[J]_r - 3\{l_a\}^T[J]_r\{l_a\}) - \tfrac{1}{2}\Omega^2\sum_{i=1}^{n}\int_{D_i}(\{l_c\}^T[l_i]^T[\hat{J}_i]_e[l_i]\{l_c\} + \mathrm{tr}([l_i]^T[\hat{J}_i]_e[l_i]) - 3\{l_a\}^T[l_i]^T[\hat{J}_i]_e[l_i]\{l_a\})\, dD_i + \tfrac{1}{2}\sum_{i=1}^{n}\int_{D_i}\rho_i\{u_{ci}\}^T[\Lambda_{1i}{}^2]\{u_{ci}\}\, dD_i \tag{19}$$

To test κ_2 for positive definiteness, we define a testing density function given by $\hat{\kappa}_2$ such that

$$\kappa_2 = \int_{D_1}\int_{D_2}\cdots\int_{D_n}\hat{\kappa}_2\, dD_1\, dD_2\ldots dD_n$$

in which D_i represents the extension of the ith domain, and test $\hat{\kappa}_2$ for positive definiteness at every point of that domain. The conditions for the positive definiteness of κ_2 thus obtained are more stringent than necessary. The sign of $\hat{\kappa}_2$ can be tested by means of Sylvester's criterion (see Ref. 9, Sec. 6.4) by writing a Hessian density matrix $[\hat{\mathscr{H}}]_E$ as the matrix of the coefficients associated with $\hat{\kappa}_2|_E$, where the latter is simply the value of $\hat{\kappa}_2$ in the neighborhood of the equilibrium point in question, namely, the origin.

In the case in which it is not possible to define density matrices, some discretization procedure may be used in conjunction with Eq. (19). One approach consists of the introduction of new time-dependent coordinates representing certain integrals involving elastic displacements. This approach may require the use of the Schwarz inequality for functions. Another discretization technique is, of course, modal analysis in conjunction with series truncation.

Stability of Motion of Earth-Pointing Satellites with Flexible Appendages

A. Method of Testing Density Functions

We shall be concerned with the stability of an Earth-pointing satellite consisting of a main rigid body with three pairs of rods (see Fig. 3), where the rods can undergo flexural motion. The rods coincide with the satellite principal axes when in undeformed state. Since in this special case axes $x_i y_i z_i$ $(i = 1,2,\ldots,6)$ and xyz coincide, it follows that $[l_i] = [1]$ $(i = 1,2,3)$ and $[l_i] = -[1]$ $(i = 4,5,6)$, where $[1]$ is the unit matrix. In view of this, $\hat{\kappa}_2$ can be written as

$$\hat{\kappa}_2 = \left(\prod_{i=1}^{6} D_i\right)^{-1}\Big[-\tfrac{1}{2}\Omega^2(\{l_c\}^T[J]_r\{l_c\} + \mathrm{tr}\,[J]_r - 3\{l_a\}^T[J]_r\{l_a\}) - \tfrac{1}{2}\Omega^2\sum_{i=1}^{6} D_i(\{l_c\}^T[\hat{J}_i]_e\{l_c\} + \mathrm{tr}\,[\hat{J}_i]_e - 3\{l_a\}^T[\hat{J}_i]_e\{l_a\}) + \tfrac{1}{2}\sum_{i=1}^{6}\rho_i D_i\{u_{ci}\}^T[\Lambda_{1i}{}^2]\{u_{ci}\}\Big] \tag{20}$$

The domains D_i $(i = 1,2,\ldots,6)$ in Eq. (20) are defined by $h_i < x_i < h_i + l_i$ $(i = 1,4)$, $h_i < y_i < h_i + l_i$ $(i = 2,5)$, $h_i < z_i < h_i + l_i$ $(i = 3,6)$. Denoting by A, B, and C the moments of inertia of the entire body in undeformed state about axes x, y and z, respectively, we can write

$$[J]_r = \begin{bmatrix} A + m(y_c^2 + z_c^2) & -mx_c y_c & -mx_c z_c \\ -mx_c y_c & B + m(x_c^2 + z_c^2) & -my_c z_c \\ -mx_c z_c & -my_c z_c & C + m(x_c^2 + y_c^2) \end{bmatrix} \tag{21}$$

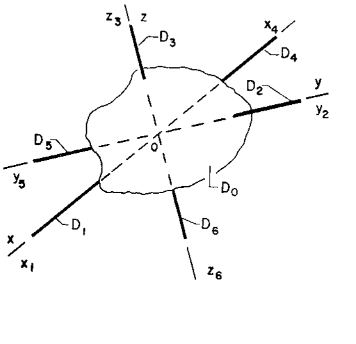

Fig. 3 The flexible Earth-pointing satellite.

Moreover, it is not difficult to show that

$$[\hat{J}_1]_e = \rho_1 \begin{bmatrix} v_{c1}^2 + w_{c1}^2 & -x_1 v_{c1} & -x_1 w_{c1} \\ -x_1 v_{c1} & w_{c1}^2 & -v_{c1} w_{c1} \\ -x_1 w_{c1} & -v_{c1} w_{c1} & v_{c1}^2 \end{bmatrix}$$

$$[\hat{J}_2]_e = \rho_2 \begin{bmatrix} w_{c2}^2 & -y_2 u_{c2} & -u_{c2} w_{c2} \\ -y_2 u_{c2} & u_{c2}^2 + w_{c2}^2 & -y_2 w_{c2} \\ -u_{c2} w_{c2} & -y_2 w_{c2} & u_{c2}^2 \end{bmatrix} \quad (22)$$

$$[\hat{J}_3]_e = \rho_3 \begin{bmatrix} v_{c3}^2 & -u_{c3} v_{c3} & -z_3 u_{c3} \\ -u_{c3} v_{c3} & u_{c3}^2 & -z_3 v_{c3} \\ -z_3 u_{c3} & -z_3 v_{c3} & u_{c3}^2 + v_{c3}^2 \end{bmatrix}$$

Matrices $[\hat{J}_4]_e$, $[\hat{J}_5]_e$, $[\hat{J}_6]_e$ are obtained from $[\hat{J}_1]_e$, $[\hat{J}_2]_e$, $[\hat{J}_3]_e$ by replacing the subscripts 1, 2, 3 by 4, 5, 6, respectively. Let us assume that system $\xi\eta\zeta$ is obtained from abc by means of the following rotations: θ_2 about b yielding the system $a'b'c'$, $-\theta_1$ about a' yielding $a''b''c''$, and θ_3 about c'' yielding $\xi\eta\zeta$. In view of this, matrices $\{l_c\}$ and $\{l_a\}$ can be shown to have the forms

$$\{l_c\} = \begin{Bmatrix} -(s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3) \\ s\theta_2 s\theta_3 - s\theta_1 c\theta_2 c\theta_3 \\ c\theta_1 c\theta_2 \end{Bmatrix}$$

$$\{l_a\} = \begin{Bmatrix} c\theta_2 c\theta_3 - s\theta_1 s\theta_2 s\theta_3 \\ -(c\theta_2 s\theta_3 + s\theta_1 s\theta_2 c\theta_3) \\ c\theta_1 s\theta_2 \end{Bmatrix} \quad (23)$$

We shall be concerned with the equilibrium configuration in which the body is at rest relative to the orbital axes, namely, that in which axes xyz and abc coincide. In an inertial space the body rotates with angular velocity Ω about z. The equilibrium configuration is defined by E: $\theta_1 = \theta_2 = \theta_3 = v_1 = w_1 = u_2 \dots = v_6 = 0$, where the latter imply also $v_{c1} = w_{c1} = u_{c2} = \dots = v_{c6} = 0$. Inserting Eqs. (21–23) into Eq. (20), and ignoring small quantities in the variables, we obtain

$$\begin{aligned} \hat{\kappa}_2|_E = \frac{1}{2l_1 l_2 \dots l_6} \{ & \Omega^2[(C-B)\theta_1^2 + 4(C-A)\theta_2^2 + 3(B-A)\theta_3^2 - \\ & 2\theta_1(\rho_2 l_2 y_2 w_{c2} + \rho_5 l_5 y_5 w_{c5} + \rho_3 l_3 z_3 v_{c3} + \rho_6 l_6 z_6 v_{c6}) - \\ & 8\theta_2(\rho_1 l_1 x_1 w_{c1} + \rho_4 l_4 x_4 w_{c4} + \rho_3 l_3 z_3 u_{c3} + \rho_6 l_6 z_6 u_{c6}) + \\ & 6\theta_3(\rho_1 l_1 x_1 v_{c1} + \rho_4 l_4 x_4 v_{c4} + \rho_2 l_2 y_2 u_{c2} + \rho_5 l_5 y_5 u_{c5})] + \\ & \rho_1 l_1[(\Lambda_{1w_1}^2 + \Omega^2) w_{c1}^2 + \Lambda_{1v_1}^2 v_{c1}^2] + \\ & \rho_4 l_4[(\Lambda_{1w_4}^2 + \Omega^2) + \Lambda_{1v_4}^2 v_{c4}^2] + \\ & \rho_2 l_2[(\Lambda_{1u_2}^2 - 3\Omega^2) u_{c2}^2 + (\Lambda_{1w_2}^2 + \Omega^2) w_{c2}^2] + \\ & \rho_5 l_5[(\Lambda_{1u_5}^2 - 3\Omega^2) + (\Lambda_{1w_5}^2 + \Omega^2) w_{c5}^2] + \\ & \rho_3 l_3[(\Lambda_{1u_3}^2 - 3\Omega^2) u_{c3}^2 + \Lambda_{1v_3}^2 v_{c3}^2] + \\ & \rho_6 l_6[(\Lambda_{1u_6}^2 - 3\Omega^2) u_{c6}^2 + \Lambda_{1v_6}^2 v_{c6}^2]\} \end{aligned} \quad (24)$$

We note that terms involing the motion x_i, y_i, z_i of the mass center explicitly have been ignored under the assumption that they are one order of magnitude lower than that of the elastic displacements. Moreover, such terms appear to second power only.

The quantity $\Lambda_{1v_1}^2$ is the lowest eigenvalue satisfying the eigenvalue problem defined by the differential equation

$$\mathscr{L}_{v_1}[v_1] = \Lambda_{v_1}^2 \rho_1 v_1 \quad (25)$$

which must be satisfied over the domain D_1, where v_1 is subject to the boundary conditions

$$\begin{aligned} B_1[v_1] = 0,\ B_2[v_1] = 0 \quad \text{at} \quad x_1 = h_1 \\ B_3[v_1] = 0,\ B_4[v_1] = 0 \quad \text{at} \quad x_1 = h_1 + l_1 \end{aligned} \quad (26)$$

Similar eigenvalue problems must be satisfied by $\Lambda_{1w_1}^2, \Lambda_{1u_2}^2, \dots, \Lambda_{1v_6}^2$. To determine the form of the operators $\mathscr{L}_{v_1}$, B_1, B_2, B_3, and B_4, we write the system elastic potential energy

$$\begin{aligned} V_{EL} = \tfrac{1}{2}\int_{D_1} \Big\{ & EI_{v_1}\left(\frac{\partial^2 v_1}{\partial x_1^2}\right)^2 + EI_{w_1}\left(\frac{\partial^2 w_1}{\partial x_4^2}\right)^2 + \\ & P_{x_1}\left[\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\frac{\partial w_1}{\partial x_1}\right)^2\right]\Big\} dx_1 + \\ \tfrac{1}{2}\int_{D_2} \Big\{ & EI_{u_2}\left(\frac{\partial^2 u_2}{\partial y_2^2}\right)^2 + EI_{w_2}\left(\frac{\partial^2 w_2}{\partial y_2^2}\right)^2 + \\ & P_{y_2}\left[\left(\frac{\partial u_2}{\partial y_2}\right)^2 + \left(\frac{\partial w_2}{\partial y_2}\right)^2\right]\Big\} dy_2 + \\ \tfrac{1}{2}\int_{D_3} & \left[EI_{u_3}\left(\frac{\partial^2 u_3}{\partial z_3^2}\right)^2 + EI_{v_3}\left(\frac{\partial^2 v_3}{\partial z_3^2}\right)^2\right] dz_3 + \dots + \\ \tfrac{1}{2}\int_{D_6} & \left[EI_{u_6}\left(\frac{\partial^2 u_6}{\partial z_6^2}\right)^2 + EI_{v_6}\left(\frac{\partial^2 v_6}{\partial z_6^2}\right)^2\right] dz_6 \end{aligned} \quad (27)$$

where EI_{v_1} is the flexural stiffness about an axis parallel to z and P_{x_1} is the axial centrifugal force at x_1. Similar definitions can be given for EI_{w_1}, EI_{u_2}, EI_{v_6}, and $P_{y_2}, \dots, P_{y_5}$. Several integrations by parts of the terms in v_1 lead us to the conclusion that the operator $\mathscr{L}_{v_1}$ has the form

$$\mathscr{L}_{v_1} = \frac{\partial^2}{\partial x_1^2}\left(EI_{v_1}\frac{\partial^2}{\partial x_1^2}\right) - \frac{\partial}{\partial x_1}\left(P_{x_1}\frac{\partial}{\partial x_1}\right) \quad (28)$$

whereas the operators defining the boundary conditions are

$$\begin{aligned} B_1 &= \frac{\partial}{\partial x_1},\ B_2 = 1,\ B_3 = EI_{v_1}\frac{\partial^2}{\partial x_1^2} \\ B_4 &= \frac{\partial}{\partial x_1}\left(EI_{v_1}\frac{\partial^2}{\partial x_1^2}\right) - P_{x_1}\frac{\partial}{\partial x_1} \end{aligned} \quad (29)$$

Moreover, the axial centrifugal force can be shown to have the expression (see Ref. 10, Sec. 10-4)

$$P_{x_1} = \tfrac{1}{2}\rho_1 \Omega^2 (h_1 + l_1)\left[1 - \left(\frac{x_1}{h_1 + l_1}\right)^2\right] \quad (30)$$

Operators $\mathscr{L}_{w_1}, \mathscr{L}_{u_2}, \dots, \mathscr{L}_{w_5}$ have the same structure as $\mathscr{L}_{v_1}$ and analogous statements can be made concerning the operators defining the corresponding boundary conditions. Operators $\mathscr{L}_{u_3}$, $\mathscr{L}_{v_3}$, $\mathscr{L}_{u_6}$, and $\mathscr{L}_{v_6}$, on the other hand, do not contain the term due to the axial force and neither do the operators B_4 associated with these vibrations.

Expression (24) can be tested for positive definiteness by means of Sylvester's criterion. To this end, we note that $\hat{\kappa}_2|_E$ can be written as the sum of three independent quadratic forms. Ignoring the common factor $(2l_1 l_2, \dots, l_6)^{-1}$, and using Sylvester's criterion, the stability conditions resulting from the requirement that all three quadratic forms be positive definite can be shown to reduce to

$$C - B - \frac{l_2(\rho_2 y_2^2)_{\max}}{[(\Lambda_{1w_2}/\Omega)^2 + 1]} - \frac{l_5(\rho_5 y_5^2)_{\max}}{[(\Lambda_{1w_5}/\Omega)^2 + 1]} - \frac{l_3(\rho_3 z_3^2)_{\max}}{(\Lambda_{1v_3}/\Omega)^2} - \frac{l_6(\rho_6 z_6^2)_{\max}}{(\Lambda_{1v_6}/\Omega)^2} > 0 \quad (31a)$$

$$C - A - \frac{4l_1(\rho_1 x_1^2)_{\max}}{[(\Lambda_{1w_1}/\Omega)^2 + 1]} - \frac{4l_4(\rho_4 x_4^2)_{\max}}{[(\Lambda_{1w_4}/\Omega)^2 + 1]} - \frac{4l_3(\rho_3 z_3^2)_{\max}}{[(\Lambda_{1u_3}/\Omega)^2 - 3]} - \frac{4l_6(\rho_6 z_6^2)_{\max}}{[(\Lambda_{1u_6}/\Omega)^2 - 3]} > 0 \quad (31b)$$

$$B - A - \frac{3l_1(\rho_1 x_1^2)_{\max}}{(\Lambda_{1v_1}/\Omega)^2} - \frac{3l_4(\rho_4 x_4^2)_{\max}}{(\Lambda_{1v_4}/\Omega)^2} - \frac{3l_2(\rho_2 y_2^2)_{\max}}{[(\Lambda_{1u_2}/\Omega)^2 - 3]} - \frac{3l_5(\rho_5 y_5^2)_{\max}}{[(\Lambda_{1u_3}/\Omega)^2 - 3]} > 0 \quad (31c)$$

$$(\Lambda_{1u_3}/\Omega)^2 > 3 \quad (31d)$$

$$(\Lambda_{1u_6}/\Omega)^2 > 3 \quad (31e)$$

$$(\Lambda_{1u_2}/\Omega)^2 > 3 \quad (31f)$$

$$(\Lambda_{1u_5}/\Omega)^2 > 3 \quad (31g)$$

since the satisfaction of these inequalities ensures that all the principal minor determinants are positive. We note that for uniform rods the quantities $(\rho_1 x_1^2)_{\max}$ $(\rho_2 y_2^2)_{\max} \dots, (\rho_6 z_6^2)_{\max}$ become $\rho_1(h_1 + l_1)^2, \rho_2(h_2 + l_2)^2, \dots, \rho_6(h_6 + l_6)^2$, respectively.

B. Method of Integral Coordinates

In contrast to the method using testing density functions, we shall now attempt to determine conditions for the positive definiteness of κ_2 by relaxing the requirement that its density be positive definite at every point of the domain of definition. To this end, we insert the results of Eqs. (21–23) into Eq. (19) and obtain the value of the κ_2 in the neighborhood of the equilibrium point in the form

$$\begin{aligned}\kappa_2|_E = \tfrac{1}{2}\{&\Omega^2[(C-B)\theta_1{}^2+4(C-A)\theta_2{}^2+3(B-A)\theta_3{}^2-\\ &2\theta_1(\textstyle\int_{D_2}\rho_2 y_2 w_{c2}\,dD_2+\int_{D_5}\rho_5 y_5 w_{c5}\,dD_5+\\ &\textstyle\int_{D_3}\rho_3 z_3 v_{c3}\,dD_3+\int_{D_6}\rho_6 z_6 v_{c6}\,dD_6)-\\ &8\theta_2(\textstyle\int_{D_1}\rho_1 x_1 w_{c1}\,dD_1+\int_{D_4}\rho_4 x_4 w_{c4}\,dD_4+\\ &\textstyle\int_{D_3}\rho_3 z_3 u_{c3}\,dD_3+\int_{D_6}\rho_6 z_6 u_{c6}\,dD_6)+\\ &6\theta_3(\textstyle\int_{D_1}\rho_1 x_1 v_{c1}\,dD_1+\int_{D_4}\rho_4 x_4 v_{c4}\,dD_4+\\ &\textstyle\int_{D_2}\rho_2 y_2 u_{c2}\,dD_2+\int_{D_5}\rho_5 y_5 u_{c5}\,dD_5)]+\\ &\textstyle\int_{D_1}\rho_1[(\Lambda_{1w_1}{}^2+\Omega^2)w_{c1}{}^2+\Lambda_{1v_1}{}^2 v_{c1}{}^2]\,dD_1+\\ &\textstyle\int_{D_4}\rho_4[(\Lambda_{1w_4}{}^2+\Omega^2)w_{c4}{}^2+\Lambda_{1v_4}v_{c4}{}^2]\,dD_4+\\ &\textstyle\int_{D_2}\rho_2[(\Lambda_{1u_2}{}^2-3\Omega^2)u_{c2}{}^2+(\Lambda_{1w_2}{}^2+\Omega^2)w_{c2}{}^2]\,dD_2+\\ &\textstyle\int_{D_5}\rho_5[(\Lambda_{1u_5}{}^2-3\Omega^2)u_{c5}{}^2+(\Lambda_{1w_5}{}^2+\Omega^2)w_{c5}{}^2]\,dD_5+\\ &\textstyle\int_{D_3}\rho_3[(\Lambda_{1u_3}{}^2-3\Omega^2)u_{c3}{}^2+\Lambda_{1v_3}{}^2 v_{c3}{}^2]\,dD_3+\\ &\textstyle\int_{D_6}\rho_6[(\Lambda_{1u_6}{}^2-3\Omega^2)u_{c6}{}^2+\Lambda_{1v_6}{}^2 v_{c6}{}^2]\,dD_6\}\end{aligned}\tag{32}$$

Note that terms of order higher than two have been neglected. Furthermore, terms involving the motions x_c, y_c, z_c of the mass center explicitly have been ignored for the same reasons as given before.

We shall now define new generalized coordinates involving integrals appearing in $\kappa_2|_E$. Specifically, let us define the following integral coordinate

$$\bar{v}_1(t) = \int_{D_1}\rho_1 x_1 v_{c1}(x_1,t)\,dD_1 \tag{33}$$

Similar definitions exist for $\bar{v}_4(t)$, $\bar{w}_1(t),\ldots,\bar{v}_6(t)$. By using Schwarz's inequality for functions, in conjunction with these integral coordinates, we convert a testing functional into a testing function whose positive definiteness ensures stability. Although this is essentially a discretization scheme, it does not involve any truncation and the results obtained by this method can be accepted with confidence.

Recalling Schwarz's inequality for functions, and concentrating on the first integral coordinate in Eq. (33), it is not difficult to show that

$$\left(\int_{D_1}\rho_1 x_1 v_{c1}\,dD_1\right)^2 \leqq \int_{D_1}\rho_1 x_1{}^2\,dD_1 \int_{D_1}\rho_1 v_{c1}{}^2\,dD_1 \tag{34}$$

Using the definition of $\bar{v}_1$, inequality (34) can be rewritten as

$$\int_{D_1}\rho_1 v_{c1}{}^2\,dD_1 \geqq \bar{v}_1{}^2/I_1 \tag{35}$$

where

$$I_1 = \int_{D_1}\rho_1 x_1{}^2\,dD_1$$

represents the moment of inertia of the rod associated with the domain D_1 about a transverse axis through 0. Similar inequalities can be derived for the integral coordinates $\bar{v}_4,\ldots,\bar{v}_6$, involving $v_{c4},\ldots,v_{c6}$, respectively. Inserting inequality (35), and the analogous inequalities for the other integral coordinates, into Eq. (32), we can define the testing function $\kappa_3|_E$ given by

$$\begin{aligned}\kappa_3|_E = \tfrac{1}{2}\{&\Omega^2[(C-B)\theta_1{}^2+4(C-A)\theta_2{}^2+3(B-A)\theta_3{}^2-\\ &2\theta_1(\bar{w}_2+\bar{w}_5+\bar{v}_3+\bar{v}_6)-8\theta_2(\bar{w}_1+\bar{w}_4+\bar{u}_3+\bar{u}_6)+\\ &6\theta_3(\bar{v}_1+\bar{v}_4+\bar{u}_2+\bar{u}_5)]+(\Lambda_{1w_1}{}^2+\Omega^2)(\bar{w}_1{}^2/I_1)+\\ &(\Lambda_{1w_4}{}^2+\Omega^2)(\bar{w}_4{}^2/I_4)+\Lambda_{1v_1}{}^2(\bar{v}_1{}^2/I_1)+\Lambda_{1v_4}{}^2(\bar{v}_4{}^2/I_4)+\\ &(\Lambda_{1u_2}{}^2-3\Omega^2)(\bar{u}_2{}^2/I_2)+(\Lambda_{1u_5}{}^2-3\Omega^2)(\bar{u}_5{}^2/I_5)+\\ &(\Lambda_{1w_2}{}^2+\Omega^2)(\bar{w}_2{}^2/I_2)+(\Lambda_{1w_5}{}^2+\Omega^2)(\bar{w}_5{}^2/I_5)+\\ &(\Lambda_{1v_3}{}^2-3\Omega^2)(\bar{u}_3{}^2/I_3)+(\Lambda_{1u_6}{}^2-3\Omega^2)(\bar{u}_6{}^2/I_6)+\\ &\Lambda_{1v_3}{}^2(\bar{v}_3{}^2/I_3)+\Lambda_{1v_6}{}^2(\bar{v}_6{}^2/I_6)\}\end{aligned}\tag{36}$$

Note, by contrast, that $\kappa_3|_E$ is a function, whereas $\kappa_2|_E$ is a functional. Due to the nature of the inequalities used, it is easy to see that

$$\kappa_3|_E \leqq \kappa_2|_E \tag{37}$$

Hence, if $\kappa_3|_E$ is positive definite, then the null solution is asymptotically stable; this can be ascertained by means of Sylvester's criterion. To this end, we form the Hessian matrix associated with $\kappa_3|_E$ and test its principal minor determinants, which leads to the following stability criteria

$$C-B-I_2[(\Lambda_{1w_2}/\Omega)^2+1]^{-1}-I_5[(\Lambda_{1w_5}/\Omega)^2+1]^{-1}- I_3(\Lambda_{1v_3}/\Omega)^{-2}-I_6(\Lambda_{1v_6}/\Omega)^{-2}>0 \tag{38a}$$

$$C-A-4I_1[(\Lambda_{1w_1}/\Omega)^2+1]^{-1}-4I_4[(\Lambda_{1w_4}/\Omega)^2+1]^{-1}- 4I_3[(\Lambda_{1u_3}/\Omega)^2-3]^{-1}-4I_6[(\Lambda_{1u_6}/\Omega)^2-3]^{-1}>0 \tag{38b}$$

$$B-A-3I_1(\Lambda_{1v_1}/\Omega)^{-2}-3I_4(\Lambda_{1v_4}/\Omega)^{-2}- 3I_2[(\Lambda_{1u_2}/\Omega)^2-3]^{-1}-3I_5[(\Lambda_{1u_5}/\Omega)^2-3]^{-1}>0 \tag{38c}$$

$$(\Lambda_{1u_3}/\Omega)^2>3 \tag{38d}$$

$$(\Lambda_{1u_6}/\Omega)^2>3 \tag{38e}$$

$$(\Lambda_{1u_2}/\Omega)^2>3 \tag{38f}$$

$$(\Lambda_{1u_5}/\Omega)^2>3 \tag{38g}$$

C. Modal Analysis

To perform a stability analysis by means of the normal modes procedure, we assume that the elastic displacements can be represented by finite series of suitable space-dependent eigenfunctions multiplying corresponding time-dependent generalized coordinates in the following form:

$$v_1 = \sum_{i=1}^{n_1}\phi_{1i}(x_1)V_{1i}(t), \qquad w_1 = \sum_{i=1}^{n_1}\psi_{1i}(x_1)W_{1i}(t) \tag{39}$$

where n_1 is a constant integer, ϕ_{1i} and ψ_{1i} are eigenfunctions corresponding to the various elastic rods, and V_{1i} and W_{1i} are associated generalized coordinates. Similar series can be expanded for $u_2, w_2,\ldots,v_6$.

Consistent with our previous discussion of the nature of the centrifugal forces, we recognize that the eigenfunctions entering into expressions (39), as well as those for $u_2, w_2,\ldots,v_6$, are defined by two distinct types of eigenvalue problems. The eigenfunctions associated with the domains D_1, D_2, D_4 and D_5 must satisfy eigenvalue problems defined by Eq. (25), where the operator $\mathscr{L}$ and the associated boundary conditions are given by Eqs. (28) and (29). On the other hand, the eigenvalues associated with the domains D_3 and D_6 again satisfy eigenvalue problems defined by Eq. (25), but the operator $\mathscr{L}$ and the associated boundary conditions do not contain terms due to the axial centrifugal force. The solution of these eigenvalue problems is discussed in Ref. 10 (see Secs. 5-10, 10-4).

The eigenfunctions possess the orthogonality property. Moreover, they can be normalized, so that

$$\int_{D_1}\rho_1\phi_{vi}(x_1)\phi_{vj}(x_1)\,dx_1 = \delta_{ij} \tag{40}$$

where δ_{ij} is Kronecker's delta. Similar expressions can be written for the remaining eigenfunctions.

In view of the preceding, if we consider the corresponding boundary conditions, a typical term in Eq. (27) becomes

$$\int_{D_1}\left[EI_{v_1}\left(\frac{\partial^2 v_1}{\partial x_1{}^2}\right)^2+P_{x_1}\left(\frac{\partial v_1}{\partial x_1}\right)^2\right]dx_1 = \sum_{i=1}^{n_1}\Lambda_{v_1 i}{}^2 V_{1i}{}^2 \tag{41}$$

Hence, the elastic potential energy V_{EL} can be regarded as a function of the generalized coordinates V_{1i}, $W_{1i},\ldots,V_{6i}$. Note that, for consistency with the first of expressions (39), the subscript v_1 and the index i in $\Lambda_{v_1 i}$ are in reverse order to that used in expressions (16) and (17).

In the normal modes procedure, it is not necessary to use the bounding properties of Rayleigh's quotient. Examining Eqs. (14) and (17), in view of the discussion following Eq. (17), we can define the new testing functional

$$\kappa_4 = -T_0+V_G+V_{EL} \tag{42}$$

which is often referred to as the dynamic potential. In this particular case, the Stability Theorem reduces to Theorem 2 of Ref. 5. Moreover, it is easy to see that if κ_4 is positive definite, H is positive definite and the system asymptotically stable. We shall assume, for the sake of this analysis, that the motion of the

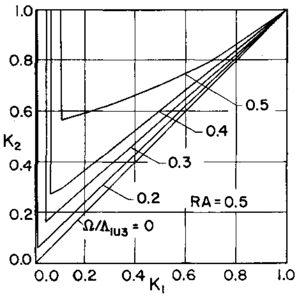

Fig. 4 Stability regions with Ω/Λ_{1u_3} as parameter.

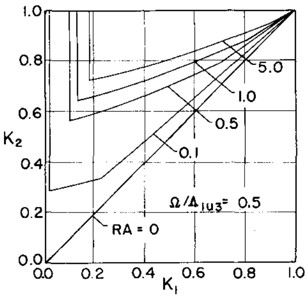

Fig. 6 Stability regions with *RA* as parameter.

mass center is negligible, which is equivalent to saying that $v_{c1} = v_1, w_{c1} = w_1$, etc. Evaluating κ_4 in the neighborhood of the equilibrium point, and replacing the elastic displacements by the appropriate modal expressions, we obtain a quadratic function in the variables $\theta_1, \theta_2, \theta_3, V_{1i}, W_{1i}, \ldots, V_{6i}$, where the function is denoted by $\kappa_4|_E$. The number of variables considered depends on the number of modes assumed, namely, on the integers $n_1, n_2, \ldots, n_6$. The sign definiteness of $\kappa_4|_E$ can be established by using Sylvester's criterion. To this end, the Hessian matrix associated with $\kappa_4|_E$ is formed and its principal minor determinants tested. This yields the following stability criteria

$$C-B-\sum_{i=1}^{n_2} J_{w2i}{}^2/(1+\Lambda_{w_{2i}}{}^2/\Omega^2)-\sum_{i=1}^{n_5} J_{w5i}{}^2/(1+\Lambda_{w_{5i}}{}^2/\Omega^2)- \sum_{i=1}^{n_3} J_{v3i}{}^2/(\Lambda_{v_{3i}}/\Omega)^2-\sum_{i=1}^{n_6} J_{v6i}{}^2/(\Lambda_{v_{6i}}/\Omega)^2 > 0 \tag{43a}$$

$$C-A-4\sum_{i=1}^{n_1} J_{w1i}{}^2/(1+\Lambda_{w_{1i}}{}^2/\Omega^2)-4\sum_{i=1}^{n_4} J_{w4i}{}^2/(1+\Lambda_{w_{4i}}/\Omega^2)- 4\sum_{i=1}^{n_3} J_{u3i}{}^2/(\Lambda_{u_{3i}}{}^2/\Omega^2-3)-4\sum_{i=1}^{n_6} J_{u6i}{}^2/(\Lambda_{u_{6i}}{}^2/\Omega^2-3) > 0 \tag{43b}$$

$$B-A-3\sum_{i=1}^{n_2} J_{u2i}{}^2/(\Lambda_{u_{2i}}{}^2/\Omega^2-3)-3\sum_{i=1}^{n_5} J_{u5i}{}^2/(\Lambda_{u_{5i}}{}^2/\Omega^2-3)- 3\sum_{i=1}^{n_3} J_{v3i}{}^2/(\Lambda_{v_{3i}}/\Omega)^2-3\sum_{i=1}^{n_6} J_{v6i}{}^2/(\Lambda_{v_{6i}}/\Omega)^2 > 0 \tag{43c}$$

$$(\Lambda_{u_{3i}}/\Omega)^2 > 3 \tag{43d}$$

$$(\Lambda_{u_{6i}}/\Omega)^2 > 3 \tag{43e}$$

$$(\Lambda_{u_{2i}}/\Omega)^2 > 3 \tag{43f}$$

$$(\Lambda_{u_{5i}}/\Omega)^2 > 3 \tag{43g}$$

where

$$J_{v1i} = \int_{h_1}^{h_1+l_1} \rho_1(x_1)\phi_{1i}(x_1)dx_1, \quad \text{etc.}$$

Numerical Results

The general solution of the stability problem of a rigid satellite with three (or less) pairs of uniform rods has been programmed for digital computation, and a numerical solution obtained. Results are presented for the criteria developed using each of the three methods considered. For the numerical study, it is assumed that the rods have equal mass and stiffness properties, equal lengths, and, in addition, the rigid body dimensions $h_1, h_2, h_3, \ldots, h_6$ are equal. The above restrictions are placed only on the numerical solution to facilitate the presentation of data; there are no such restrictions placed on either the problem formulation, or computer program. Figure 4 shows the regions of stability for a rigid satellite in the parameter plane defined by K_1 and K_2, where $K_1 = (C-B)/A$ and $K_2 = (C-A)/B$. We notice that this plot is the same as that given in Ref. 11, which considers a strictly rigid satellite, except for the change in sign in the definition of K_2, and for the fact that the "Delp" region has not been included here because it is not Liapunov stable. The results of Ref. 11 appear in Fig. 4 as the special case defined by $\Omega/\Lambda_{1u_3} = 0$. To verify this statement, we note that if we consider the rods as rigid members the associated natural frequencies become infinite, and the terms introduced into the criteria by the elastic effects reduce to zero. The results shown in Figs. 4–6 are based on criteria derived by using the method of integral coordinates. In Fig. 4 the effect of the spin ratio Ω/Λ_{1u_3} is shown. Curves of constant spin ratio are plotted for a fixed value of the parameter RA, where $RA = 2I/A_0$. Note that A_0 is the moment of inertia associated with the rigid domain D_0. The regions of stability lie above and to the right of corresponding bounding curves. We note that as the frequency ratio increases the size of the stability region considered decreases. In fact, examining inequality (38d), we see that for a frequency ratio $\Omega/\Lambda_{1u^3} = (1/3)^{1/2}$ the motion is unstable, regardless of the value of RA. In Fig. 4 we notice that the curves of constant Ω/Λ_{1u_3} have abrupt slope changes, the reason for these changes being demonstrated in Fig. 5. We recall from our analysis that three separate criteria involving the moments of inertia must be satisfied for stability. The abrupt changes in the slope of the stability boundary in Fig. 4 occur at the points of intersection of the stability boundaries obtained using each of the three criteria. Figure 6 shows the effect on system stability of changes in the parameter RA. We notice that increasing RA decreases the size of the region of stability. Figure 7 shows a comparison of the three methods presented. The curve representing the modal analysis was obtained using a three mode approximation. The figure shows close agreement between the criteria obtained by the modal analysis and those obtained by the method of integral coordinates. The normal mode approximation yields a slightly larger region of stability, as should be expected. The number of modes used did not seem to affect the results appreciably and, indeed almost identical results were obtained using one mode or two through five modes. This, however, can

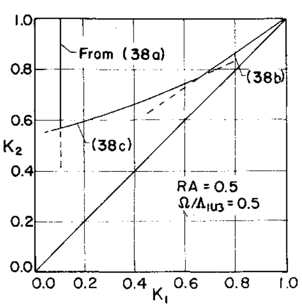

Fig. 5 Detailed plot of stability boundaries.

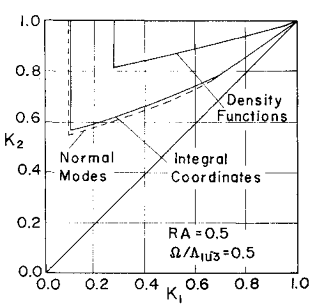

Fig. 7 Comparative stability boundaries plots.

be attributed to the type of deformation obtained for this particular problem, due to the relatively low spin involved, and should not be regarded as a clue as to the number of modes to be used for any other problem. The comparison between the method of integral coordinates and that using testing density functions shows that the criteria obtained by using testing density functions are much more restrictive than those derived by using integral coordinates. This difference is due to the fact that the method using testing density functions bases the stability criteria on the worst possible case as far as moments of inertia of the elastic appendages are concerned, whereas the method of integral coordinates bases them on certain average values that involve quantities directly proportional to moments of inertia of these appendages. The difference in the stability regions obtained by the two methods tends to be largest when the mass of the rods is concentrated near the points of attachment and reduces to zero for the case in which the mass of the rods is concentrated at the free ends.

Summary and Conclusions

This paper presents three different approaches to the Liapunov stability analysis of hybrid dynamical systems, namely, the method based on testing density functions, the method of integral coordinates, and the modal analysis. The first two of these represent new developments, whereas the last one has been used by many authors, including those of this paper, in conjunction with this and various other problems. Also in contrast to the modal analysis, the first two methods yield closed-form stability criteria, but in general the criteria derived by the method of integral coordinates are less restrictive than those derived by using testing density functions. On the other hand, the method using testing density functions is easier to apply and requires less work than the method of integral coordinates. Moreover, whereas the definition of a testing density function is generally straightforward, in cases other than that considered here the definition of integral coordinates may cause difficulties. The modal analysis generally yields more involved stability criteria, depending on the number of modes used to represent the elastic displacements. This, of course, implies that the solution of the eigenvalue problem associated with the elastic parts is known. For elastic domains of peculiar shape or nonuniform stiffness and mass distributions the solution of the eigenvalue problem alone may cause difficulties.

All three methods can be used to determine stability regions in the parameter space. However, the modal analysis is considerably more laborious than the method using testing density functions or integral coordinates. Moreover, the uncertainty concerning the modal truncation effect on the stability criteria still remains, and further work on this subject is desirable.

References

[1] Meirovitch, L., "Stability of a Spinning Body Containing Elastic Parts via Liapunov's Direct Method," *AIAA Journal*, Vol. 8, No. 7, July 1970, pp. 1193–1200.

[2] Meirovitch, L. and Nelson, H. D., "On the High-Spin Motion of a Satellite Containing Elastic Parts," *Journal of Spacecraft and Rockets*, Vol. 3, No. 11, Nov. 1966, pp. 1597–1602.

[3] Dokuchaev, L. V., "Plotting the Regions of Stable Rotation of a Space Vehicle with Elastic Rods," *Kosmicheskie Issledovaniya* (English translation), Vol. 7, No. 4, July-Aug. 1969, pp. 534–546.

[4] Meirovitch, L., "A Method for the Liapunov Stability Analysis of Force-Free Hybrid Dynamical Systems," *AIAA Journal*, Vol. 9, No. 9, Sept. 1971, pp. 1695–1701.

[5] Meirovitch, L., "Liapunov Stability Analysis of Hybrid Dynamical Systems with Multi-Elastic Domains," *International Journal of Non-Linear Mechanics*, Vol. 7, 1972, pp. 425–443.

[6] Meirovitch, L. and Calico, R., "The Stability of Motion of Force-Free Spinning Satellites with Flexible Appendages," *Journal of Spacecraft and Rockets*, Vol. 9, No. 4, April 1972, pp. 237–245.

[7] Meirovitch, L. and Calico, R., "The Stability of Motion of Satellites with Flexible Appendages," CR-1978, Feb. 1972, NASA.

[8] Moiseyev, N. N. and Rumyanstev, V. V., *Dynamic Stability of Bodies Containing Fluid*, Springer-Verlag, New York, 1968.

[9] Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill, New York, 1970.

[10] Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, New York, 1967.

[11] DeBra, D. B. and Delp, R. H., "Rigid Body Attitude Stability and Natural Frequencies in a Circular Orbit," *Journal of the Astronautical Sciences*, Vol. 8, 1961, pp. 14–17.